

COFREE COALGEBRAS OVER OPERADS II. HOMOLOGY INVARIANCE

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ABSTRACT. This paper gives conditions under which the cofree coalgebras constructed in [11] are homology invariant.

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1. INTRODUCTION

The paper [11] constructed cofree coalgebras over operads cogenerated by free chain-complexes over a ring R . The underlying chain-complexes of these cofree coalgebras were not known to be free in the case where $R = \mathbb{Z}$ since they were only submodules of the Baer-Specker group, \mathbb{Z}^{\aleph_0} — see [4] for a survey of this group.

In the present paper we address several issues:

- (1) We extend the construction of cofree coalgebras to the class of nearly free modules — see definition 2.1 and appendix A. This class includes free modules but is closed under the operations of

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taking countable products and cofree coalgebras. Consequently, it will be possible to iterate our cofree coalgebra construction.

- (2) We show that, under fairly weak conditions on the operad — that it is composed of projective modules that are finitely generated in each dimension — cofree coalgebras of nearly free chain-complexes are homology invariant.

Section 2 defines nearly free modules and other terms connected with operads and coalgebras over them.

Section 3 carries out step 1 above. It essentially shows that cofree coalgebras preserve direct limits. Since nearly free modules are direct limits of free modules, this defines cofree coalgebras over nearly free modules.

Section 4 shows that cofibrant operads are homotopy functors — i.e. homotopies of maps induce homotopies of cofree coalgebra morphisms. This, coupled with the results of appendix C implies that they preserve homology equivalences of nearly free chain-complexes.

Our main result, proved in section 5 is:

Corollary 5.7: Let R be a field or \mathbb{Z} and let $\mathcal{V} = \{\mathcal{V}(n)\}$ be an operad such that $\mathcal{V}(n)$ is RS_n -projective and finitely generated in each dimension for all $n > 0$. If

$$W_{\mathcal{V}}C = \left\{ \begin{array}{c} L_{\mathcal{V}}C \\ M_{\mathcal{V}}C \\ P_{\mathcal{V}}C \\ \mathcal{F}_{\mathcal{V}}C \end{array} \right\}$$

— the cofree coalgebras defined in [11] — and

$$f: C \rightarrow D$$

is a homology equivalence of nearly free chain complexes (see definition 2.1) that are bounded from below, then the induced map

$$W_{\mathcal{V}}f: W_{\mathcal{V}}C \rightarrow W_{\mathcal{V}}D$$

is a homology equivalence.

Remark 1.1. The condition on \mathcal{V} is essentially equivalent to the condition of being Σ -cofibrant in [2].

This condition is necessary because there are well-known cases in which it does not hold and the associated cofree coalgebras are *not* homology invariant.

2. DEFINITIONS

Throughout this paper, R will denote a field or \mathbb{Z} .

Definition 2.1. An R -module M will be called *nearly free* if every countable submodule is R -free.

Remark. This condition is automatically satisfied unless $R = \mathbb{Z}$.

Clearly, any \mathbb{Z} -free module is also nearly free. The Baer-Specker group, \mathbb{Z}^{\aleph_0} , is a well-known example of a nearly free \mathbb{Z} -module that is *not* free — see [5], [1], and [12]. Compare this with the notion of \aleph_1 -free groups — see [3].

By abuse of notation, we will often call chain-complexes nearly free if their underlying modules are (when one ignores grading).

Nearly free \mathbb{Z} -modules enjoy useful properties that free modules do *not*. For instance, in many interesting cases, the cofree coalgebra of a nearly free chain-complex is nearly free.

We will denote the closed symmetric monoidal category of (not necessarily free) R -chain-complexes with R -tensor products by $\mathbf{Ch}(R)$. These chain-complexes are allowed to extend into arbitrarily many negative dimensions and have underlying graded R -modules that are

- arbitrary if R is a field (but they will be free)
- *nearly free*, in the sense of definition 2.1, if $R = \mathbb{Z}$.

Definition 2.2. The object $I \in \mathbf{Ch}(R)$, the *unit interval*, is defined by

$$I_k = \begin{cases} R \cdot p_0 \oplus R \cdot p_1 & \text{if } k = 0 \\ R \cdot q & \text{if } k = 1 \\ 0 & \text{if } k \neq 0, 1 \end{cases}$$

where p_0, p_1, q are just names for the canonical generators of I , and the one nonzero boundary map is defined by $q \mapsto p_1 - p_0$.

We also define, for any object $A \in \mathbf{Ch}(R)$, the *cone on A* , denoted \bar{A} and equal to $A \otimes I / A \otimes R \cdot p_1$. There are canonical morphisms $A \rightarrow \bar{A}$ and $\bar{A} \rightarrow \Sigma A$, where $\Sigma: \mathbf{Ch}(R) \rightarrow \mathbf{Ch}(R)$ is the functor that raises the grading by 1.

Two morphisms

$$f_0, f_1: C \rightarrow D$$

in $\mathbf{Ch}(R)$, are defined to be *chain-homotopic* if there exists a morphism

$$F: C \otimes I \rightarrow D$$

such that $F|_{C \otimes R \cdot p_i} = f_i: C \rightarrow D$. This is well-known to be equivalent to the existence of a degree +1 map $\Phi: C \rightarrow D$ such that $\partial_D \circ \Phi + \Phi \circ \partial_C = f_1 - f_0$.

We make extensive use of the Koszul Convention (see [6]) regarding signs in homological calculations:

Definition 2.3. If $f: C_1 \rightarrow D_1$, $g: C_2 \rightarrow D_2$ are maps, and $a \otimes b \in C_1 \otimes C_2$ (where a is a homogeneous element), then $(f \otimes g)(a \otimes b)$ is defined to be $(-1)^{\deg(g) \cdot \deg(a)} f(a) \otimes g(b)$.

Remark 2.4. If f_i, g_i are maps, it isn't hard to verify that the Koszul convention implies that $(f_1 \otimes g_1) \circ (f_2 \otimes g_2) = (-1)^{\deg(f_2) \cdot \deg(g_1)} (f_1 \circ f_2 \otimes g_1 \circ g_2)$.

Definition 2.5. Given chain-complexes $A, B \in \mathbf{Ch}(R)$ define

$$\mathrm{Hom}_R(A, B)$$

to be the chain-complex of graded R -morphisms where the degree of an element $x \in \mathrm{Hom}_R(A, B)$ is its degree as a map and with differential

$$\partial f = f \circ \partial_A - (-1)^{\deg f} \partial_B \circ f$$

As a R -module $\mathrm{Hom}_R(A, B)_k = \prod_j \mathrm{Hom}_R(A_j, B_{j+k})$.

Remark. Given $A, B \in \mathbf{Ch}(R)^{S_n}$, we can define $\mathrm{Hom}_{RS_n}(A, B)$ in a corresponding way.

Definition 2.6. Define:

- (1) Set_f to be the category of finite sets and bijections. Let Set_f^2 be the category of finite sets whose elements are also finite sets. Morphisms are bijections of sets that respect the “fine structure” of elements that are also sets. There is a *forgetful functor*

$$\mathfrak{f}: \mathrm{Set}_f^2 \rightarrow \mathrm{Set}_f$$

that simply forgets that the elements of an object of Set_f^2 are, themselves, finite sets. There is also a “*flattening*” functor

$$\mathfrak{g}: \mathrm{Set}_f^2 \rightarrow \mathrm{Set}_f$$

that sends a set (of sets) to the union of the elements (regarded as sets).

- (2) For a finite set X , $\Sigma_X = \mathrm{End}_{\mathrm{Set}_f}(X)$.
- (3) $\mathrm{Set}_f\text{-mod}$ to be the category of contravariant functors $\mathrm{Func}(\mathrm{Set}_f^{\mathrm{op}}, \mathbf{Ch}(R))$, with morphisms that are natural transformations.
- (4) Given $C, D \in \mathrm{Set}_f\text{-mod}$, define $\underline{\mathrm{Hom}}(C, D)$ to be the set of natural transformations of functors. Also define $\underline{\mathrm{Hom}}_X(C, D)$, where $X \in \mathrm{Set}_f$, to be the natural transformations of C and D restricted to sets isomorphic to X (i.e., of the same cardinality). Both of these functors are chain-complexes.
- (5) $\Sigma\text{-mod}$ to be the category of sequences $\{M(n)\}$, $m \geq 1$ where $M(n) \in \mathbf{Ch}(R)$ and $M(n)$ is equipped with a right S_n -action.

Remark. If $[n]$ is the set of the first n positive integers, then $\Sigma_{[n]} = S_n$, the symmetric group. If M is a Set_f -module then, for each finite set, X , there is a right Σ_X -action on $M(X)$.

We follow the convention that $S_0 = S_1 = \{1\}$, the trivial group.

Note that $\Sigma\text{-mod}$ is what is often called the category of collections.

If $\mathbf{a} = \{\{x\}, \{y, z, t\}, \{h\}\} \in \text{Set}_f^2$ then $f(\mathbf{a}) \cong [3]$, a set of three elements, and $g(\mathbf{a}) = \{x, y, z, t, h\}$.

It is well-known that the categories $\text{Set}_f\text{-mod}$ and $\Sigma\text{-mod}$ are isomorphic — see section 1.7 in part I of [9]. The restriction isomorphism

$$r: \text{Set}_f\text{-mod} \rightarrow \Sigma\text{-mod}$$

simply involves evaluating functors on the finite sets $[n]$ for all $n \geq 1$. If $F \in \text{Set}_f\text{-mod}$, then $r(F) = \{F([n])\}$. The functorial nature of F implies that $F([n])$ is equipped with a natural S_n -action. The functors $\underline{\text{Hom}}_n(C, D)$ correspond to $\text{Hom}_{RS_n}(C([n]), D([n]))$ and the fact that morphisms in Set_f preserve cardinality imply that

$$\underline{\text{Hom}}(C, D) = \prod_{n \geq 0} \underline{\text{Hom}}_n(C, D)$$

Although Set_f -modules are equivalent to modules with a symmetric group action, it is often easier to formulate operadic constructions in terms of $\text{Set}_f\text{-mod}$. Equivariance relations are automatically satisfied.

Definition 2.7. If X is a finite set of cardinality n the *set of orderings of X* is

$$\text{Ord}(X) = \{f|f: X \xrightarrow{\cong} [n]\}$$

Now we define a Set_f analogue to the multiple tensor product. Given a set X of cardinality n , and an assignment of an object $C_x \in \mathbf{Ch}(R)$ for each element $x \in X$, we can define, for each $g \in \text{Ord}(X)$ a product

$$\bigotimes_g C_x = C_{g^{-1}(1)} \otimes \cdots \otimes C_{g^{-1}(n)}$$

The symmetry of tensor products determines a morphism

$$\bar{\sigma}: \bigotimes_g C_x \rightarrow \bigotimes_{\sigma \circ g} C_x$$

for each $\sigma \in S_n$ which essentially permutes factors and multiplies by ± 1 , following the Koszul Convention in definition 2.3.

Definition 2.8. The *unordered tensor product* is defined by

$$\bigotimes_X C_x = \text{coequalizer}_{\sigma \in S_n} \left\{ \bar{\sigma}: \bigoplus_{g \in \text{Ord}(X)} \bigotimes_g C_x \rightarrow \bigoplus_{g \in \text{Ord}(X)} \bigotimes_g C_x \right\}$$

If $C \in \mathbf{Ch}(R)$ and $X \in \mathbf{Set}_f$ then C^X will denote the unordered tensor product

$$\bigotimes_X C$$

of copies of C indexed by elements of X , and C^\otimes will denote the \mathbf{Set}_f -module whose value on $X \in \mathbf{Set}_f$ is C^X .

We use $X \cdot C$ to denote a direct sum of n copies of C , where n is the cardinality of a finite set X .

When $X \in \mathbf{Set}_f^2$,

$$\bigotimes_X C$$

is regarded as being taken over $f(X)$ — i.e., we “forget” that the elements of X are sets themselves.

Remark. The unordered tensor product is isomorphic (as an object of $\mathbf{Ch}(R)$) to the tensor product of the C_x , as x runs over the elements of X . The coequalizer construction determines how the it behaves with respect to *set-morphisms*.

If $X = [n]$, then $C^{[n]} = C^n$. Note that $C^X \otimes C^Y = C^{X \sqcup Y}$, for $X, Y \in \mathbf{Set}_f$. We also follow the convention that $C^\emptyset = \mathbb{1} = R$, concentrated in dimension 0.

Definition 2.9. If $X \in \mathbf{Set}_f$, $x \in X$ and $\{f_y: V_y \rightarrow U_y\}$ are morphisms of $\mathbf{Ch}(R)$ indexed by elements $y \in X$ then define

$$\bigotimes_{X,x} (U, V) = \bigotimes_{y \in X} Z_y \xrightarrow{1 \otimes \dots \otimes f_x \otimes \dots \otimes 1} \bigotimes_{y \in X} U_y$$

to be the unordered tensor product, where

$$Z_y = \begin{cases} U_y & \text{if } y \neq x \\ V_y & \text{if } y = x \end{cases}$$

Remark. Given any ordering of the elements of the set X , there exists a canonical isomorphism

$$\bigotimes_{X,x} (U, V) = \underbrace{U \otimes \dots \otimes V \otimes \dots \otimes U}_{\text{position } x}$$

Definition 2.10. Let $X, Y \in \mathbf{Set}_f$ and let $x \in X$. Define

$$X \sqcup_x Y = (X \setminus \{x\}) \cup Y$$

Remark. Note that $X \sqcup_x \emptyset = X \setminus \{x\}$.

Proposition. If $X, Y, Z \in \text{Set}_f$, and $xx_1, X_2 \in X$ and $y \in Y$, then

$$\begin{aligned} X \sqcup_x (Y \sqcup_y Z) &= (X \sqcup_x Y) \sqcup_y Z \\ (X \sqcup_{x_1} Y) \sqcup_{x_2} Z &= (X \sqcup_{x_2} Z) \sqcup_{x_1} Y \end{aligned}$$

Definition 2.11. An *operad* in $\mathbf{Ch}(R)$ is a Set_f -module, C equipped with operations

$$\circ_x: C(X) \otimes C(Y) \rightarrow C(X \sqcup_x Y)$$

for all $x \in X$ and all $X, Y \in \text{Set}_f$ and satisfying the two axioms

(1) *Associativity*:

$$\circ_x(1 \otimes \circ_y) = \circ_y(\circ_x \otimes 1):$$

$$C(X) \otimes C(Y) \otimes C(Z) \rightarrow C(X \sqcup_x (Y \sqcup_y Z))$$

$$\circ_{x_2}(\circ_{x_1} \otimes 1) = \circ_{x_1}(\circ_{x_2} \otimes 1)(1 \otimes \tau):$$

$$C(X) \otimes C(Y) \otimes C(Z) \rightarrow C((X \sqcup_{x_1} Y) \sqcup_{x_2} Z)$$

for all $X, Y, Z \in \text{Set}_f$ and all $xx_1, x_2 \in X$ and $y \in Y$, where $\tau: C(Y) \otimes C(Z) \rightarrow C(Z) \otimes C(Y)$ is the transposition isomorphism.

(2) *Unit*: There exist morphisms $\eta_x: \mathbb{1} \rightarrow C(\{x\})$ for all singleton sets $\{x\} \in \text{Set}_f$ that make the diagrams

$$\begin{array}{ccc} C(X) \otimes \mathbb{1} & \xrightarrow{\cong} & C(X) \\ 1 \otimes \eta_x \downarrow & \nearrow \circ_x & \\ C(X) \otimes C(x) & & \end{array} \quad \begin{array}{ccc} \mathbb{1} \otimes C(X) & \xrightarrow{\cong} & C(X) \\ \eta_x \otimes 1 \downarrow & \nearrow \circ_x & \\ C(X) & & \end{array}$$

commute, for all $X \in \text{Set}_f$. The operad will be called *nonunital* if the axioms above only hold for *nonempty* sets.

Remark. See theorem 1.60 and 1.61 and section 1.7.1 of [9] for the proof that this defines operads correctly. For more traditional definitions, see [11], [7]. This is basically the definition of a pseudo-operad in [9] where we have added the unit axiom. To translate this definition into the more traditional ones, set the n^{th} component of the operad to $C([n])$.

The use of $\text{Set}_f\text{-mod}$ causes the equivariance conditions in [7] to be automatically satisfied.

The operads we consider here correspond to *symmetric* operads in [11].

The term “unital operad” is used in different ways by different authors. We use it in the sense of Kriz and May in [7], meaning the operad has a 0-component that acts like an arity-lowering augmentation under compositions. This is $C(\emptyset) = \mathbb{1}$.

A simple example of an operad is:

Example 2.12. For each finite set, X , $C(X) = \mathbb{Z}\Sigma_X$, with composition defined by inclusion of sets. This operad is denoted \mathfrak{S}_0 . In other notation, its n^{th} component is the *symmetric group-ring* $\mathbb{Z}S_n$.

For the purposes of this paper, the canonical example of an operad is

Definition 2.13. Given any $C \in \mathbf{Ch}(R)$, the associated *coendomorphism operad*, $\text{CoEnd}(C)$ is defined by

$$\text{CoEnd}(C)(X) = \text{Hom}_R(C, C^X)$$

for $X \in \text{Set}_f$, and $C^X = \bigotimes_X C$ is the unordered tensor product defined in definition 2.8. The compositions $\{\circ_x\}$ are defined by

$$\begin{aligned} \circ_x: \text{Hom}_R(C, C^X) \otimes \text{Hom}_R(C, C^Y) &\rightarrow \\ \text{Hom}_R(C, C^{X \setminus \{x\}} \otimes C_x \otimes \text{Hom}_R(C, C^Y)) &\xrightarrow{\text{Hom}_R(1, 1 \otimes e)} \\ \text{Hom}_R(C, C^{X \setminus \{x\}} \otimes C^Y) &= \text{Hom}_R(C, C^{X \sqcup_x Y}) \end{aligned}$$

where C_x is the copy of C corresponding to $x \in X$ and $e: C_x \otimes \text{Hom}_R(C, C^Y) \rightarrow C^Y$ is the evaluation morphism. This is a non-unital operad, but if $C \in \mathbf{Ch}(R)$ has an augmentation map $\varepsilon: C \rightarrow \mathbb{1}$ then we can set

$$\text{CoEnd}(C)(\emptyset) = \mathbb{1}$$

and

$$\begin{aligned} \circ_x: \text{Hom}_R(C, C^X) \otimes \text{Hom}_R(C, C^\emptyset) &= \text{Hom}_R(C, C^X) \otimes \mathbb{1} \\ &\xrightarrow{\text{Hom}_R(1, 1_{X \setminus \{x\}} \otimes \varepsilon_x)} \text{Hom}_R(C, C^{X \setminus \{x\}}) \end{aligned}$$

where $1_{X \setminus \{x\}}: C^{X \setminus \{x\}} \rightarrow C^{X \setminus \{x\}}$ is the identity map and $\varepsilon_x: C_x \rightarrow \mathbb{1}$ is the augmentation, applied to the copy of C indexed by $x \in X$.

Given $C \in \mathbf{Ch}(R)$ with *subcomplexes* $\{D_1, \dots, D_k\}$, the *relative coendomorphism operad* $\text{CoEnd}(C; \{D_i\})$ is defined to be the sub-operad of $\text{CoEnd}(C)$ consisting of maps $f \in \text{Hom}_R(C, C^X)$ such that $f(D_j) \subseteq D_j^X \subseteq C^X$ for all j .

We use the coendomorphism operad to define the main object of this paper:

Definition 2.14. A *coalgebra over an operad* \mathcal{V} is a chain-complex $C \in \mathbf{Ch}(R)$ with an operad morphism $\alpha: \mathcal{V} \rightarrow \text{CoEnd}(C)$, called its *structure map*. We will sometimes want to define coalgebras using the *adjoint structure map*

$$\bar{\alpha}: C \rightarrow \underline{\text{Hom}}(\mathcal{V}, C^\otimes)$$

(in $\mathbf{Ch}(R)$) or even the set of chain-maps

$$\bar{\alpha}_X: C \rightarrow \underline{\mathrm{Hom}}_X(\mathcal{V}(X), C^X)$$

for all $X \in \mathrm{Set}_f$.

We can also define the analogue of an ideal:

Definition 2.15. Let C be a coalgebra over the operad \mathcal{U} with adjoint structure map

$$\alpha: C \rightarrow \underline{\mathrm{Hom}}(\mathcal{U}, C^\otimes)$$

and let $D \subseteq [C]$ be a sub-chain complex that is a direct summand. Then D will be called a *coideal* of C if the composite

$$\alpha|_D: D \rightarrow \underline{\mathrm{Hom}}(\mathcal{U}, C^\otimes) \xrightarrow{\underline{\mathrm{Hom}}(1_{\mathcal{U}}, p^\otimes)} \underline{\mathrm{Hom}}(\mathcal{U}, (C/D)^\otimes)$$

vanishes, where $p: C \rightarrow C/D$ is the projection to the quotient (in $\mathbf{Ch}(R)$).

Remark. Note that it is easier for a sub-chain-complex to be a coideal of a coalgebra than to be an ideal of an algebra. For instance, all sub-coalgebras of a coalgebra are also coideals. Consequently it is easy to form quotients of coalgebras and hard to form sub-coalgebras. This is dual to what occurs for algebras.

We will sometimes want to focus on a particular class of \mathcal{V} -coalgebras: the *pointed, irreducible coalgebras*. We define this concept in a way that extends the conventional definition in [13]:

Definition 2.16. Given a coalgebra over a unital operad \mathcal{V} with adjoint structure-map

$$a_X: C \rightarrow \underline{\mathrm{Hom}}_X(\mathcal{V}(X), C^X)$$

an element $c \in C$ is called *group-like* if $a_X(c) = f_X(c^X)$ for all $n > 0$. Here $c^X \in C^X$ is the n -fold R -tensor product, where n is the cardinality of X ,

$$f_X = \mathrm{Hom}_R(\epsilon_X, 1): \mathrm{Hom}_R(1, C^X) = C^X \rightarrow \underline{\mathrm{Hom}}_X(\mathcal{V}(X), C^X)$$

and $\epsilon_X: \mathcal{V}(X) \rightarrow \mathcal{V}(\emptyset) = 1 = R$ is the augmentation (which is n -fold composition with $\mathcal{V}(\emptyset)$).

A coalgebra C over an operad \mathcal{V} is called *pointed* if it has a *unique* group-like element (denoted 1), and *pointed irreducible* if the intersection of any two sub-coalgebras contains this unique group-like element.

Remark. Note that a group-like element generates a sub \mathcal{V} -coalgebra of C and must lie in dimension 0.

Although this definition seems contrived, it arises in “nature”: The chain-complex of a pointed, simply-connected reduced simplicial set is naturally a pointed irreducible coalgebra over the Barratt-Eccles operad, $\mathfrak{S} =$

$\{C(K(S_n, 1))\}$ (see [10]). In this case, the operad action encodes the chain-level effect of Steenrod operations.

Proposition 2.17. *Let D be a pointed, irreducible coalgebra over an operad \mathcal{V} . Then the augmentation map*

$$\varepsilon: D \rightarrow R$$

is naturally split and any morphism of pointed, irreducible coalgebras

$$f: D_1 \rightarrow D_2$$

is of the form

$$1 \oplus \bar{f}: D_1 = R \oplus \ker \varepsilon_{D_1} \rightarrow D_2 = R \oplus \ker \varepsilon_{D_2}$$

where $\varepsilon_i: D_i \rightarrow R$, $i = 1, 2$ are the augmentations.

Proof. The definition (2.16) of the sub-coalgebra $R \cdot 1 \subseteq D_i$ is stated in an invariant way, so that any coalgebra morphism must preserve it. Any morphism must also preserve augmentations because the augmentation is the 0th-order structure-map. Consequently, f must map $\ker \varepsilon_{D_1}$ to $\ker \varepsilon_{D_2}$. The conclusion follows. \square

Definition 2.18. We denote the *category* of coalgebras over \mathcal{V} by \mathcal{S}_0 . If \mathcal{V} is unital, every \mathcal{V} -coalgebra, C , comes equipped with a canonical augmentation

$$\varepsilon: C \rightarrow R$$

so the *terminal object* is R . If \mathcal{V} is not unital, the terminal object in this category is 0, the null coalgebra.

The category of *pointed irreducible coalgebras* over \mathcal{V} is denoted \mathcal{S}_0 — this is only defined if \mathcal{V} is unital. Its terminal object is the coalgebra whose underlying chain complex is R concentrated in dimension 0.

We also need:

Definition 2.19. If $A \in \mathcal{C} = \mathcal{S}_0$ or \mathcal{S}_0 , then $\lceil A \rceil$ denotes the underlying chain-complex in $\mathbf{Ch}(R)$ of

$$\ker A \rightarrow t$$

where t denotes the terminal object in \mathcal{C} — see definition 2.18. We will call $\lceil * \rceil$ the *forgetful functor* from \mathcal{C} to $\mathbf{Ch}(R)$.

We will use the concept of cofree coalgebra cogenerated by a chain complex:

Definition 2.20. Let D be a coalgebra over an operad \mathcal{U} , equipped with a $\mathbf{Ch}(R)$ -morphism $\varepsilon: \lceil D \rceil \rightarrow E$, where $E \in \mathbf{Ch}(R)$. Then D is called *the cofree coalgebra over \mathcal{U} cogenerated by ε* if any morphism in $\mathbf{Ch}(R)$

$$f: \lceil C \rceil \rightarrow E$$

where C is a \mathcal{U} -coalgebra, induces a *unique* morphism of \mathcal{U} -coalgebras

$$\alpha_f: C \rightarrow D$$

that makes the diagram

$$\begin{array}{ccc} [C] & \xrightarrow{[\alpha_f]} & [D] \\ & \searrow f & \downarrow \varepsilon \\ & & E \end{array}$$

commute. Here α_f is called the *classifying map* of f . If C is a \mathcal{U} -coalgebra then

$$\alpha_1: C \rightarrow L_{\mathcal{U}}[C]$$

will be called the *classifying map* of C .

This universal property of cofree coalgebras implies that they are unique up to isomorphism if they exist.

3. EXTENDING THE CONSTRUCTION IN [11]

The paper [11] gave an explicit construction of $L_{\mathcal{U}}C$ when C was an R -free chain complex. When R is a field, all chain-complexes are R -free, so the results of the present paper are already true in that case.

Consequently, we will restrict ourselves to the case where $R = \mathbb{Z}$.

Proposition 3.1. *The forgetful functor (defined in definition 2.19) and cofree coalgebra functors define adjoint pairs*

$$\begin{aligned} P_{\mathcal{V}}(*): \mathbf{Ch}(R) &\rightleftarrows \mathcal{I}_0: [*] \\ L_{\mathcal{V}}(*): \mathbf{Ch}(R) &\rightleftarrows \mathcal{S}_0: [*] \end{aligned}$$

Remark. The adjointness of the functors follows from the universal property of cofree coalgebras — see [11].

The Adjoints and Limits Theorem in [8] implies that:

Theorem 3.2. *If $\{A_i\}$ is an inverse system in $\mathbf{Ch}(R)$ and $\{C_i\}$ is a direct system in \mathcal{I}_0 or \mathcal{S}_0 then*

$$\begin{aligned} \varprojlim P_{\mathcal{V}}(A_i) &= P_{\mathcal{V}}(\varprojlim A_i) \\ \varprojlim L_{\mathcal{V}}(A_i) &= L_{\mathcal{V}}(\varprojlim A_i) \\ [\varinjlim C_i] &= \varinjlim [C_i] \end{aligned}$$

Remark. This implies that *direct* limits in \mathcal{I}_0 or \mathcal{S}_0 are the same as direct limits of underlying chain-complexes.

Proposition 3.3. *If $C \in \mathbf{Ch}(R)$, let $\mathcal{G}(C)$ denote the lattice of countable subcomplexes of C . Then*

$$C = \varinjlim \mathcal{G}(C)$$

Proof. Clearly $\varinjlim \mathcal{G}(C) \subseteq C$ since all of the canonical maps to C are inclusions. Equality follows from every element $x \in C$ being contained in a finitely generated subcomplex of C consisting of x and $\partial(x)$. \square

Lemma 3.4. *Let $n > 1$ be an integers, let F be a finitely-generated projective (non-graded) $\mathbb{Z}S_n$ -module, and let $\{C_\alpha\}$ a direct system of modules. Then the natural map*

$$\varinjlim \mathrm{Hom}_{RS_n}(F, C_\alpha) \rightarrow \mathrm{Hom}_{RS_n}(F, \varinjlim C_\alpha)$$

is an isomorphism.

If F and the $\{C_\alpha\}$ are graded, the corresponding statement is true if F is finitely-generated and $\mathbb{Z}S_n$ -projective in each dimension.

Proof. We will only prove the non-graded case. The graded case follows from the fact that the maps of the $\{C_\alpha\}$ preserve grade.

In the non-graded case, finite generation of F implies that the natural map

$$\bigoplus_{\alpha} \mathrm{Hom}_{RS_n}(F, C_\alpha) \rightarrow \mathrm{Hom}_{RS_n}(F, \bigoplus_{\alpha} C_\alpha)$$

is an isomorphism. The projectivity of F implies that $\mathrm{Hom}_{RS_n}(F, *)$ is exact, so the short exact sequence defining the direct limit is preserved. \square

Theorem 3.5. *Let $\mathcal{V} = \{\mathcal{V}(X)\}$ be an operad and let C be a chain-complex with $\mathcal{G}(C) = \{C_\alpha\}$ the direct system of countable subcomplexes ordered by inclusion. In addition, suppose:*

- (1) *For all $n \geq 0$, $\mathcal{V}(X)$ is $\mathbb{Z}\Sigma_X$ -projective and finitely generated in each dimension.*
- (2) *C is nearly free (see definition 2.1).*

Then the cofree coalgebras

$$L_{\mathcal{V}}C, P_{\mathcal{V}}C, M_{\mathcal{V}}C, \mathcal{F}_{\mathcal{V}}C$$

are well-defined and

$$\left. \begin{aligned} L_{\mathcal{V}}C &= \varinjlim L_{\mathcal{V}}C_\alpha \\ P_{\mathcal{V}}C &= \varinjlim P_{\mathcal{V}}C_\alpha \\ M_{\mathcal{V}}C &= \varinjlim M_{\mathcal{V}}C_\alpha \\ \mathcal{F}_{\mathcal{V}}C &= \varinjlim \mathcal{F}_{\mathcal{V}}C_\alpha \end{aligned} \right\} \subseteq \underline{\mathrm{Hom}}(\mathcal{V}, C^{\otimes})$$

Remark. Indeed, the construction of them given in [11] is valid in this case.

Proof. The only part of the construction in [11] that uses \mathbb{Z} -freeness is the proof that the $L_{\mathcal{V}}C$ are coalgebras — i.e., that the diagrams in Appendix B of [11] commute. The construction of the $L_{\mathcal{V}}C$ (as *chain-complexes*) does not use it.

The near-freeness of C implies that the C_{α} are all free.

We will regard the *chain-complex*, $[L_{\mathcal{V}}C]$, as the result of this construction in Lemma 3.4 of [11] — setting aside questions of whether it's a coalgebra.

We have $C = \varinjlim C_{\alpha}$ and the conditions on \mathcal{V} (and lemma 3.4) imply that

$$\underline{\mathrm{Hom}}(\mathcal{V}, C^{\otimes}) = \prod_{n>0} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n) = \prod_{n>0} \varinjlim \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C_{\alpha_n}^n)$$

where the C_{α_n} are countable. This and the \mathbb{Z} -flatness of C implies that every

$$x \in \prod_{n>0} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n)$$

lies in the image of

$$\prod_{n>0} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C_{\alpha_n}^n)$$

for some countable subcomplexes $\{C_{\alpha_n}\}$.

We claim the natural map

$$\varinjlim [L_{\mathcal{V}}C_{\alpha}] \rightarrow [L_{\mathcal{V}}C]$$

is surjective. If $x \in [L_{\mathcal{V}}C]$ is contained in

$$\prod_{n>0} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C_{\alpha_n}^n) \subseteq \prod_{n>0} \mathrm{Hom}_{RS_n}(\mathcal{V}_n, C^n)$$

where the $\{C_{\alpha_n}\}$ are all countable, then

$$\bar{C} = \sum_{n=1}^{\infty} C_{\alpha_n}$$

is also countable, and x is in the image of an element $y \in [L_{\mathcal{V}}\bar{C}]$.

Consequently

$$[L_{\mathcal{V}}C] = \varinjlim [L_{\mathcal{V}}C_{\alpha}]$$

and theorem 3.2 implies that this direct limit has a natural coalgebra structure. The conclusion follows. \square

4. COFIBRANT OPERADS

We define conditions on operads that ensure they are homotopy functors and then apply the main result to show that they are homology invariant.

Now we determine the conditions necessary to make cofree coalgebras into homotopy functors.

The relative coendomorphism operad of the unit interval is

Condition 4.1. *Throughout the rest of this section, we assume that \mathcal{V} is an operad equipped with a morphism of operads*

$$\delta: \mathcal{V} \rightarrow \mathcal{V} \otimes \mathfrak{S}_0$$

(see definition 2.13) that makes the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\delta} & \mathcal{V} \otimes \mathfrak{S}_0 \\ & \searrow & \downarrow \\ & & \mathcal{V} \end{array}$$

commute. Here, the operad structure on $\mathcal{V} \otimes \mathfrak{S}_0$ is just the tensor product of the operad structures of \mathcal{V} and \mathfrak{S}_0 .

We also assume that the arity-1 component of \mathcal{V} is equal to R , generated by the unit.

This is similar to the conditions satisfied by Σ -split operads in [2]. It is also satisfied by cofibrant operads — of which the most straightforward class is that of free operads.

The significance of this condition is given by:

Proposition 4.2. *Suppose the operad \mathcal{V} satisfies Condition 4.1 and C is a \mathcal{V} -coalgebra. Then the coalgebra structure of C naturally extends to a coalgebra structure on $C \otimes I$ whose restrictions to $C \otimes p_i$, $i = 0, 1$ agree with the coalgebra structure of C .*

Proof. The results of appendix B imply that

$$\mathrm{CoEnd}(I; \{\mathbb{Z} \cdot p_0, \mathbb{Z} \cdot p_1\}) = \mathfrak{S}_0$$

so that Condition 4.1 implies that the operad morphism

$$\mathcal{V} \rightarrow \mathrm{CoEnd}(C)$$

defining the coalgebra structure of C , lifts to a morphism

$$\mathcal{V} \rightarrow \mathrm{CoEnd}(C) \otimes \mathrm{CoEnd}(I; \{\mathbb{Z} \cdot p_0, \mathbb{Z} \cdot p_1\}) \rightarrow \mathrm{CoEnd}(C \otimes I)$$

whose coalgebra-structure on $C \otimes \{p_i\}$, $i = 0, 1$ coincides with that of C . \square

Our condition implies that:

Proposition 4.3. *Let C and D be objects of $\mathbf{Ch}(R)$ and let*

$$f_1, f_2: C \rightarrow D$$

be chain-homotopic morphisms via a chain-homotopy

$$(4.1) \quad F: C \otimes I \rightarrow D$$

Then the induced maps

$$\begin{aligned} P_V f_i: P_V C &\rightarrow P_V D \\ L_V f_i: L_V C &\rightarrow L_V D \end{aligned}$$

$i = 1, 2$, are left-homotopic in \mathcal{S}_0 and \mathcal{S}_0 , respectively via a chain homotopy

$$F': P_V f_i: (P_V C) \otimes I \rightarrow P_V D$$

If we equip $C \otimes I$ with a coalgebra structure using condition 4.1 and proposition 4.2, and if F in 4.1 is a coalgebra morphism, then the diagram

$$\begin{array}{ccc} C \otimes I & \xrightarrow{F} & D \\ \alpha_C \otimes 1 \downarrow & & \downarrow \alpha_D \\ P_V(C) \otimes I & \xrightarrow{F'} & P_V D \end{array}$$

commutes in the pointed irreducible case and the diagram

$$\begin{array}{ccc} C \otimes I & \xrightarrow{F} & D \\ \alpha_C \otimes 1 \downarrow & & \downarrow \alpha_D \\ L_V(C) \otimes I & \xrightarrow{F'} & L_V D \end{array}$$

commutes in the general case. Here α_C and α_D are classifying maps of coalgebra structures.

Proof. We will prove this in the pointed irreducible case. The general case follows by a similar argument. The chain-homotopy between the f_i induces

$$P_V F: P_V(C \otimes I) \rightarrow P_V D$$

Now we construct the map

$$H: (P_V C) \otimes I \rightarrow P_V(C \otimes I)$$

using the universal property of a cofree coalgebra and the fact that the coalgebra structure of $(P_V C) \otimes I$ extends that of $P_V C$ on both ends by condition 4.1. Clearly

$$P_V F \circ H: (P_V C) \otimes I \rightarrow P_V D$$

is the required left-homotopy.

If we define a coalgebra structure on $C \otimes I$ using condition 4.1, we get diagram

$$\begin{array}{ccccc}
 C \otimes I & \xlongequal{\quad} & C \otimes I & \xrightarrow{F} & D \\
 \alpha_C \otimes 1 \downarrow & & \downarrow \alpha_{C \otimes I} & & \downarrow \alpha_D \\
 P_{\mathcal{V}}(C) \otimes I & \xrightarrow{H} & P_{\mathcal{V}}(C \otimes I) & \xrightarrow{P_{\mathcal{V}}F} & P_{\mathcal{V}}D \\
 \epsilon_C \otimes 1 \downarrow & & \downarrow \epsilon_{C \otimes I} & & \\
 C \otimes I & \xlongequal{\quad} & C \otimes I & &
 \end{array}$$

where $\alpha_{C \otimes I}$ is the classifying map for the coalgebra structure on $C \otimes I$.

We claim this diagram commutes. The fact that F is a coalgebra morphism implies that the upper right square commutes. The large square on the left (bordered by $C \otimes I$ on all four corners) commutes by the property of co-augmentation maps and classifying maps. The two smaller squares on the left (i.e., the large square with the map H added to it) commute by the universal properties of cofree coalgebras (which imply that induced maps to cofree coalgebras are uniquely determined by their composites with co-augmentations). The diagram in the statement of the result is just the outer upper square of this diagram, so we have proved the claim. \square

Theorem 4.4. *Let \mathcal{V} be a cofibrant operad whose n^{th} component is $\mathbb{Z}S_n$ -projective and finitely generated for all $n > 0$, and let*

$$f: C \rightarrow D$$

be a homology equivalence of nearly free chain-complexes that are bounded from below. Then the induced morphisms

$$\begin{aligned}
 L_{\mathcal{V}}f: L_{\mathcal{V}}C &\rightarrow L_{\mathcal{V}}D \\
 M_{\mathcal{V}}f: M_{\mathcal{V}}C &\rightarrow M_{\mathcal{V}}D \\
 P_{\mathcal{V}}f: P_{\mathcal{V}}C &\rightarrow P_{\mathcal{V}}D \\
 \mathcal{F}_{\mathcal{V}}f: \mathcal{F}_{\mathcal{V}}C &\rightarrow \mathcal{F}_{\mathcal{V}}D
 \end{aligned}$$

are homology equivalences.

Proof. This is a direct application of lemma C.1, where

$$F = H_*(\text{suitable cofree coalgebra functor})$$

Here, we have used the fact that cofibrant operads automatically satisfy condition 4.1. \square

5. THE GENERAL CASE

This section states and proves theorem 5.6.

We can relativize the definition of cofree coalgebra in definition 2.20:

Definition 5.1. Let $f: \mathcal{U} \rightarrow \mathcal{V}$ be a morphism of operads and let $C \in \mathbf{Ch}(R)$. Any \mathcal{V} -coalgebra, A , can be pulled back over f to a \mathcal{U} -coalgebra, f^*A . The *relative cofree coalgebra with respect to the morphism f and cogenerated by C* , denoted $L_f C$ solves the universal problem:

Given any \mathcal{V} -coalgebra, A , and any morphism in $\mathbf{Ch}(R)$ $g: [f^*A] \rightarrow C$, there exists a unique morphism of \mathcal{U} -coalgebras $\hat{g}: f^*A \rightarrow L_f C$ that makes the diagram

$$\begin{array}{ccc} f^*A & \xrightarrow{\hat{g}} & L_f C \\ & \searrow g & \downarrow \varepsilon \\ & & C \end{array}$$

commute. Here, the map $\varepsilon: L_f C \rightarrow C$ is the cogeneration map.

Remark. These “not so cofree” coalgebras are universal targets of the subclass of \mathcal{U} -coalgebras that have been pulled back over f . In like fashion, we can define $M_f C$, $P_f C$, and $\mathcal{F}_f C$.

The universal property of $L_f C$ immediately implies that:

Proposition 5.2. *Under the hypotheses of definition 5.1*

$$L_f C = \alpha_\varepsilon(f^* L_{\mathcal{V}} C) \subseteq L_{\mathcal{U}} C$$

where $\alpha_\varepsilon: f^* L_{\mathcal{V}} C \rightarrow L_{\mathcal{U}} C$ is the canonical morphism of the \mathcal{U} -coalgebra, $f^* L_{\mathcal{V}} C$ to $L_{\mathcal{U}} C$ induced by the cogeneration-projection $\varepsilon: [f^* L_{\mathcal{V}} C] \rightarrow C$ (see definition 2.20).

Remark 5.3. Corresponding statements clearly hold for $M_f C$, $P_f C$, and $\mathcal{F}_f C$. The morphism $\alpha_\varepsilon: f^* L_{\mathcal{V}} C \rightarrow L_{\mathcal{U}} C$ is not usually injective.

The main idea used in theorem 5.6 is contained in:

Lemma 5.4. *Let $C \in \mathbf{Ch}(R)$ be nearly free, let \mathcal{H} be a projective operad that is finitely generated in each dimension, and let $\iota: \mathcal{J} \hookrightarrow \mathcal{H}$ be the inclusion of an operadic ideal, inducing the map*

$$\underline{\mathbf{Hom}}(\iota, 1): \underline{\mathbf{Hom}}(\mathcal{H}, C^\otimes) \rightarrow \underline{\mathbf{Hom}}(\mathcal{J}, C^\otimes)$$

If K is the kernel of the composite

$$\kappa: [L_{\mathcal{H}} C] \xrightarrow{p} \underline{\mathbf{Hom}}(\mathcal{H}, C^\otimes) \xrightarrow{\underline{\mathbf{Hom}}(\iota, 1)} \underline{\mathbf{Hom}}(\mathcal{J}, C^\otimes)$$

where

$$p: C \oplus \underline{\mathbf{Hom}}(\mathcal{H}, C^\otimes) \rightarrow \underline{\mathbf{Hom}}(\mathcal{H}, C^\otimes)$$

is the projection, then K is the pullback of a coalgebra over \mathcal{H}/\mathcal{J} via the projection

$$\mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$$

that satisfies the universal requirements for being the cofree coalgebra $L_{\mathcal{H}/\mathcal{J}}C$.

Proof. See appendix D for the proof. \square

We can prove corresponding statements for the truncated and pointed-irreducible cofree coalgebras:

Corollary 5.5. *Under the hypotheses of lemma 5.4, if M is the kernel of the composite*

$$[M_{\mathcal{H}}C] \xrightarrow{p} \underline{\text{Hom}}(\mathcal{H}, C^{\otimes}) \xrightarrow{\underline{\text{Hom}}(\mathfrak{u}, 1)} \underline{\text{Hom}}(\mathcal{J}, C^{\otimes})$$

where $k = 0$ if \mathcal{H} is unital and 1 otherwise, then $M = \text{im } M_{\mathcal{H}/\mathcal{J}}C$ in $M_{\mathcal{H}}C$ under the natural map induced by the projection $\mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$.

If \mathcal{H} is a unital operad and P is the kernel of the composite

$$[P_{\mathcal{H}}C] \xrightarrow{p} \underline{\text{Hom}}(\mathcal{H}, C^{\otimes}) \xrightarrow{\underline{\text{Hom}}(\mathfrak{u}, 1)} \underline{\text{Hom}}(\mathcal{J}, C^{\otimes})$$

then $P = \text{im } P_{\mathcal{H}/\mathcal{J}}C \subseteq P_{\mathcal{H}}C$. If \mathcal{F} is the kernel of the composite

$$[\mathcal{F}_{\mathcal{H}}C] \xrightarrow{p} \underline{\text{Hom}}(\mathcal{H}, C^{\otimes}) \xrightarrow{\underline{\text{Hom}}(\mathfrak{u}, 1)} \underline{\text{Hom}}(\mathcal{J}, C^{\otimes})$$

and \mathcal{H}/\mathcal{J} is a unital operad, then $\mathcal{F} = \text{im } \mathcal{F}_{\mathcal{H}/\mathcal{J}}C \subseteq \mathcal{F}_{\mathcal{H}}C$.

Proof. The proof of lemma 5.4 does not use any specific property of $L_{\mathcal{H}}C$ other than the facts that

- (1) it is a coalgebra that is a submodule of $\underline{\text{Hom}}(\mathcal{H}, C^{\otimes})$
- (2) its coproduct is dual to the compositions of \mathcal{H}
- (3) it is cofree in a suitable context

It is only necessary to remark that the fact that \mathcal{H}/\mathcal{J} is unital implies that $\eta(1) \notin \mathcal{J}_1$ so that the basepoint of $P_{\mathcal{H}}C$ and $\mathcal{F}_{\mathcal{H}}C$ lie in P and \mathcal{F} , respectively. \square

Now we define functoriality of cofree coalgebras with respect to operad-morphisms:

Theorem 5.6. *Let $f: \mathcal{J} \hookrightarrow \mathcal{H}$ be the inclusion of an operadic ideal with \mathcal{H} a projective free operad, $\mathcal{V} = \mathcal{H}/\mathcal{J}$ a projective operad, and with canonical projection $p: \mathcal{H} \rightarrow \mathcal{H}/\mathcal{J} = \mathcal{V}$. In addition, let $C \in \mathbf{Ch}(R)$ be nearly free. Then the kernels of*

$$\left\{ \begin{array}{l} \alpha_{\mathcal{E}}: f^* L_{\mathcal{H}}C \rightarrow L_{\mathcal{J}}C \\ \alpha_{\mathcal{E}}: f^* M_{\mathcal{H}}C \rightarrow M_{\mathcal{J}}C \\ \alpha_{\mathcal{E}}: f^* P_{\mathcal{H}}C \rightarrow P_{\mathcal{J}}C \\ \alpha_{\mathcal{E}}: f^* \mathcal{F}_{\mathcal{H}}C \rightarrow \mathcal{F}_{\mathcal{J}}C \end{array} \right\}$$

(see definition 2.20 and proposition 5.2 for an explanation of the notation α_ϵ) are

$$\left\{ \begin{array}{l} [p^*L_{\mathcal{V}}C]/C \\ [p^*M_{\mathcal{V}}C]/C \\ [p^*P_{\mathcal{V}}C]/R \\ [p^*\mathcal{F}_{\mathcal{V}}C]/R \end{array} \right\}$$

respectively. If

$$W_*C = \left\{ \begin{array}{l} L_*C \\ M_*C \\ P_*C \\ \mathcal{F}_*C \end{array} \right\}$$

then $W_fC \subseteq W_{\mathcal{J}}C$ has the structure of an \mathcal{H} -coalgebra. This coalgebra structure induces an \mathcal{H} -coalgebra morphism $\hat{f}: W_fC \rightarrow W_{\mathcal{H}}C$ that is a right inverse to f^* . This, in turn, induces a splitting of underlying chain-complexes

$$(5.1) \quad [W_{\mathcal{H}}C] \cong [W_{\mathcal{V}}C]/C \oplus [W_fC]$$

If \mathcal{H} is finitely generated in each dimension any homology equivalence

$$C \rightarrow C'$$

of nearly free modules that are bounded from below induces a homology equivalence

$$W_{\mathcal{V}}C \rightarrow W_{\mathcal{V}}C'$$

Remark. Note that $W_fC = \alpha_\epsilon(f^*W_{\mathcal{H}}C) \subseteq L_{\mathcal{J}}C$, by proposition 5.2 and remark 5.3.

This result's key ideas can be summarized as follows:

- (1) An operad morphism $f: \mathcal{U} \rightarrow \mathcal{V}$ induces a \mathcal{U} -coalgebra morphism

$$f^*: W_{\mathcal{V}}C \rightarrow W_{\mathcal{U}}C$$

whose kernel is a priori a *coideal* (see definition 2.15).

- (2) In the special case where $f: \mathcal{J} \rightarrow \mathcal{H}$ is the inclusion of an operadic ideal, the kernel of the induced map

$$f^*: W_{\mathcal{H}}C \rightarrow W_{\mathcal{J}}C$$

is a full-fledged \mathcal{H} -coalgebra (this is the main thrust of appendix D), hence the image of f^* is also an \mathcal{H} -coalgebra (being the quotient of two such).

- (3) The universal property of a cofree coalgebra implies the existence of a *unique* map

$$c: f^*(W_{\mathcal{H}}C) \rightarrow W_{\mathcal{H}}C$$

splitting f^* .

- (4) This universal property also implies that the kernel is the pullback of $W_{\mathcal{H}/\mathcal{J}}C$.
- (5) The homology invariance of $W_{\mathcal{H}}C$ — established in theorem 4.4 — implies that of $W_{\mathcal{H}/\mathcal{J}}C$.

Proof. We will prove this in the case where $W_*C = L_*C$. The other cases follow by similar arguments.

Lemma 5.4 implies that the kernel, Z , of

$$\alpha_{\mathcal{E}}: f^*L_{\mathcal{H}}C \rightarrow L_{\mathcal{J}}C$$

is the pullback of a coalgebra over $\mathcal{V} = \mathcal{H}/\mathcal{J}$. Here, the fact that \mathcal{J} is an operadic ideal implies that Z is a *sub-coalgebra* rather than a mere coideal — indeed, it is $p^*L_{\mathcal{V}}C$. The subcoalgebra $Z \oplus C \subseteq L_{\mathcal{H}}C$ (where C is equipped with a coproduct that is identically 0) is also the pullback of a coalgebra over \mathcal{V} and has the universal property of $p^*L_{\mathcal{V}}C$ so $Z \oplus C = p^*L_{\mathcal{V}}C$. Consider

$$\varepsilon_{\mathcal{V}}: p^*L_{\mathcal{V}}C \rightarrow C$$

where C is regarded as a \mathcal{V} -coalgebra whose coproduct is *identically zero*. The kernel of $\varepsilon_{\mathcal{V}}$ will be a *coideal* in $p^*L_{\mathcal{V}}C$ (see definition 2.15) whose underlying chain complex is isomorphic to $[p^*L_{\mathcal{V}}C]/C$ (since $C \subset p^*L_{\mathcal{V}}C$ is a direct summand as a chain complex and as a coalgebra).

We claim that $\ker \varepsilon_{\mathcal{V}}$ is *also* a coideal in $L_{\mathcal{H}}C$. Consider the diagram

$$\begin{array}{ccc}
 \ker \varepsilon_{\mathcal{V}} & \searrow & \\
 \downarrow & & \searrow \\
 \underline{\mathrm{Hom}}(\mathcal{H}, (p^*L_{\mathcal{V}}C)^{\otimes}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C)^{\otimes}) \\
 \downarrow & & \downarrow \\
 \underline{\mathrm{Hom}}(\mathcal{H}, (p^*L_{\mathcal{V}}C/\ker \varepsilon_{\mathcal{V}})^{\otimes}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C/\ker \varepsilon_{\mathcal{V}})^{\otimes})
 \end{array}$$

where the maps from $\ker \varepsilon_{\mathcal{V}}$ are the structure maps of $p^*L_{\mathcal{V}}C$ and $L_{\mathcal{H}}C$ and the remaining downward maps are induced by projection to the quotient. The upper triangle commutes since $p^*L_{\mathcal{V}}C = Z \oplus C$ is a sub-coalgebra of $L_{\mathcal{H}}C$. The remaining squares commute by naturality of projection to the quotient.

The composite of the vertical maps on the left is 0 because $\ker \varepsilon_{\mathcal{V}}$ is a coideal in $p^*L_{\mathcal{V}}C$ — see definition 2.15. The commutativity of the diagram implies that the composite of the vertical maps on the right is also 0, so $\ker \varepsilon_{\mathcal{V}}$ is a coideal in $L_{\mathcal{H}}C$.

It follows that the quotient

$$L_fC = L_{\mathcal{H}}C/\ker \varepsilon_{\mathcal{V}} \subseteq L_{\mathcal{J}}C$$

is an \mathcal{H} -coalgebra equipped with a canonical cogeneration (chain-)map

$$\varepsilon_f: [L_f C] \rightarrow C$$

This chain-map and the universal property of the cofree coalgebra $L_{\mathcal{H}} C$ implies the existence of a coalgebra morphism

$$\hat{f}: L_f C \rightarrow L_{\mathcal{H}} C$$

The composite of this with α_ε is a morphism that covers the identity map of C — which must be the identity map of $L_f C \subseteq L_{\mathcal{H}} C$ due to the uniqueness of induced maps to cofree coalgebras. Consequently, \hat{f} splits α_ε and induces the splitting of chain-complexes in equation 5.1.

The final statements follows from lemma 5.4 and the fact that every operad is the surjective image of some free operad. So the splitting in equation 5.1 exists for any \mathcal{V} and suitable free operad. This splitting induces a corresponding splitting in homology

$$H_*([L_{\mathcal{H}} C]) \cong H_*([L_{\mathcal{V}} C]/C) \oplus H_*([L_f C])$$

The statement about homology invariance of $L_{\mathcal{V}} C$ follows from theorem 4.4 and the fact that a direct summand of an isomorphism is an isomorphism. \square

Corollary 5.7. *Let R be a field or \mathbb{Z} and let $\mathcal{V} = \{\mathcal{V}(n)\}$ be an operad such that $\mathcal{V}(n)$ is RS_n -projective and finitely generated in each dimension for all $n > 0$. If*

$$W_{\mathcal{V}} C = \left\{ \begin{array}{c} L_{\mathcal{V}} C \\ M_{\mathcal{V}} C \\ P_{\mathcal{V}} C \\ \mathcal{F}_{\mathcal{V}} C \end{array} \right\}$$

and

$$f: C \rightarrow D$$

is a homology equivalence of nearly free chain complexes (see definition 2.1) that are bounded from below, then the induced map

$$W_{\mathcal{V}} f: W_{\mathcal{V}} C \rightarrow W_{\mathcal{V}} D$$

is a homology equivalence.

Proof. Given \mathcal{V} satisfying the hypotheses, let \mathcal{H} be the free operad generated by the components of \mathcal{V} . It will satisfy the hypotheses of theorem 5.6 and there will exist a canonical surjection of operads

$$\mathcal{H} \rightarrow \mathcal{V}$$

whose kernel is an operadic ideal. \square

APPENDIX A. NEARLY FREE MODULES

In this section, we will explore the class of nearly free \mathbb{Z} -modules — see definition 2.1. We show that this is closed under the operations of taking direct sums, tensor products, countable products and cofree coalgebras. It appears to be fairly large, then, and it would be interesting to have a direct algebraic characterization.

Clearly a module must be torsion-free (hence flat) to be nearly free. The converse is not true, however: \mathbb{Q} is flat but *not* nearly free.

The definition immediately implies that:

Proposition A.1. *Any submodule of a nearly free module is nearly free.*

Nearly free modules are closed under operations that preserve free modules:

Proposition A.2. *Let M and N be \mathbb{Z} -modules. If they are nearly free, then so are $M \oplus N$ and $M \otimes N$.*

Infinite direct sums of nearly free modules are nearly free.

Proof. If $F \subseteq M \oplus N$ is countable, so are its projections to M and N , which are free by hypothesis. It follows that F is a countable submodule of a free module.

The case where $F \subseteq M \otimes N$ follows by a similar argument: The elements of F are finite linear combinations of monomials $\{m_\alpha \otimes n_\alpha\}$ — the set of which is countable. Let

$$\begin{aligned} A &\subseteq M \\ B &\subseteq N \end{aligned}$$

be the submodules generated, respectively, by the $\{m_\alpha\}$ and $\{n_\alpha\}$. These will be countable modules, hence \mathbb{Z} -free. It follows that

$$F \subseteq A \otimes B$$

is a free module.

Similar reasoning proves the last statement, using the fact that any direct sum of free modules is free. \square

Proposition A.3. *Let $\{F_n\}$ be a countable collection of \mathbb{Z} -free modules. Then*

$$\prod_{n=1}^{\infty} F_n$$

is nearly free.

Proof. In the case where $F_n = \mathbb{Z}$ for all n

$$B = \prod_{n=1}^{\infty} \mathbb{Z}$$

is the Baer-Specker group, which is well-known to be nearly free — see [1], [5, vol. 1, p. 94 Theorem 19.2], and [3]. It is also well-known *not* to be \mathbb{Z} -free — see [12] or the survey [4].

In the general case,

$$\prod_{n=1}^{\infty} F_n$$

is a direct sum of copies of B , which is nearly free by proposition A.2. \square

Corollary A.4. *Let $\{N_k\}$ be a countable set of nearly free modules. Then*

$$\prod_{k=1}^{\infty} N_k$$

is also nearly free.

Proof. Let

$$F \subset \prod_{k=1}^{\infty} N_k$$

be countable. If F_k is its projection to factor N_k , then F_k will be countable, hence free. It follows that

$$F \subset \prod_{k=1}^{\infty} F_k$$

and the conclusion follows from proposition A.3. \square

Corollary A.5. *Let A be nearly free and let F be \mathbb{Z} -free of countable rank. Then*

$$\mathrm{Hom}_{\mathbb{Z}}(F, A)$$

is nearly free.

Proof. This follows from corollary A.4 and the fact that

$$\mathrm{Hom}_{\mathbb{Z}}(F, A) \cong \prod_{k=1}^{\mathrm{rank}(F)} A$$

\square

Corollary A.6. *Let $\{F_n\}$ be a sequence of $\mathbb{Z}S_n$ -projective modules and let A be nearly free. Then*

$$\prod_{n=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}S_n}(F_n, A^n)$$

is nearly free.

Proof. This is a direct application of the results of this section and the fact that

$$\mathrm{Hom}_{\mathbb{Z}S_n}(F_n, A^n) \subseteq \mathrm{Hom}_{\mathbb{Z}}(F_n, A^n) \subseteq \mathrm{Hom}_{\mathbb{Z}}(\hat{F}_n, A^n)$$

where \hat{F}_n is a $\mathbb{Z}S_n$ -free module of which F_n is a direct summand. \square

Theorem A.7. *Let C be a nearly free \mathbb{Z} -module and let \mathcal{V} be an operad whose n^{th} component is $\mathbb{Z}S_n$ -projective and finitely generated for all n . Then*

$$\begin{aligned} &[L_{\mathcal{V}}C] \\ &[M_{\mathcal{V}}C] \\ &[P_{\mathcal{V}}C] \\ &[\mathcal{F}_{\mathcal{V}}C] \end{aligned}$$

are all nearly free.

Proof. This follows from theorem 3.5 which states that all of these are submodules of

$$\prod_{n=1}^{\infty} \mathrm{Hom}_{\mathbb{Z}S_n}(\mathcal{V}_n, A^n)$$

and the fact that near-freeness is inherited by submodules. \square

APPENDIX B. THE RELATIVE COENDOMORPHISM OPERAD OF THE UNIT INTERVAL

Our main result is:

Proposition B.1. *If I is the unit interval (see definition 2.2), its relative coendomorphism operad (see definition 2.13) is given by*

$$\mathrm{CoEnd}(I; \{\mathbb{Z} \cdot p_0, \mathbb{Z} \cdot p_1\}) = \mathfrak{S}_0$$

defined in Example 2.12.

Proof. We must compute homomorphisms

$$g: I \rightarrow I^n$$

that send the endpoints $\{p_0, p_1\}$ to the subcomplex of I^n generated by tensor products of the endpoints — i.e.

$$\mathbb{Z}p_0 \otimes \cdots \otimes p_0 \oplus \mathbb{Z}p_1 \otimes \cdots \otimes p_1$$

Both of these subcomplexes (of I and I^n) are concentrated in dimension 0, which implies that all of our maps must be of degree zero.

It follows that all components of $\mathrm{CoEnd}(I; \{\mathbb{Z} \cdot p_0, \mathbb{Z} \cdot p_1\})$ are concentrated in dimension 0. Chain-maps of I are determined by where they send the 1-dimensional element, q . Thus we want chain-maps

$$g: I \rightarrow I^n$$

with $\partial(g(q)) = p_1 \otimes \cdots \otimes p_1 - p_0 \otimes \cdots \otimes p_0$ (n factors in each term).

We use a “geometric argument.” Consider the unit cube in \mathbb{R}^n with coordinates

$$0 \leq x_i \leq 1$$

for $i = 1, \dots, n$. Regard the edges of this as 1-simplices and the vertices as 0-simplices. Chains with the required property correspond to sequences of these 1-simplices forming paths along the edges of the cube from $(0, \dots, 0)$ to $(1, \dots, 1)$.

We claim there are exactly $n!$ such paths and they are linearly independent chains in $C(I^n)_1$. To construct a path, one must travel 1 unit in the x_i direction, then 1 unit in the $x_{i'}$ direction, with $i' \neq i$, and so on. One represents this by a list of n distinct integers between 1 and n :

$$(i, i', \dots)$$

Such lists clearly correspond to permutations $\sigma \in S_n$:

$$(\sigma(1), \sigma(2), \dots, \sigma(n))$$

Let $\{v_0, \dots, v_n\}$ be coordinates of the vertices one encounters during this process with $v_0 = (0, \dots, 0)$ and $v_n = (1, \dots, 1)$.

Since $v_{k+1} - v_k$ determines the direction one went in the k^{th} step (and since each path travels in a direction taken by no other in *some* step), it follows that each path has a *vertex* not contained in any other. This implies that each path also has a *1-simplex* not contained in any other. Consequently the paths represent linearly independent chains of $C(I^n)_1$.

It is also clear that the symmetric group permutes these $n!$ paths by permuting coordinate axes. This demonstrates a natural equality

$$\text{CoEnd}(I; \{\mathbb{Z} \cdot p_0, \mathbb{Z} \cdot p_1\})([n]) = \mathbb{Z}S_n$$

□

APPENDIX C. HOMOTOPY AND DIRECT LIMITS

This section’s main result may be summed up by the phrase

“A homotopy functor that commutes with direct limits is a homology functor of nearly free complexes.”

Let $\mathbf{K}(\mathbb{Z})$ denote the *chain-homotopy category* of \mathbb{Z} -chain-complexes — compare to the notation in § 20.4 of [14]. Objects in this category are chain-homotopy equivalence classes of chain complexes (not necessarily torsion free) and chain-homotopy morphisms are equivalent.

We have the related category, $\mathbf{D}(\mathbb{Z})$ — essentially the Verdier derived category of \mathbb{Z} . Its objects are chain-complexes where homology equivalent complexes are considered equivalent (the Verdier derived category considered cochain complexes).

We also consider the subcategory $\mathbf{K}_{\text{cell}} \subseteq \mathbf{K}(\mathbb{Z})$ of cellular chain-complexes — Exercise 10.4.5 of [14]. These are chain complexes

$$C = \bigcup_{i=1}^{\infty} C_i$$

where C_{i+1}/C_i is \mathbb{Z} -free and has vanishing differential. Clearly, any \mathbb{Z} -free chain complex that is bounded from below is in this category.

We use the following well-known properties of \mathbf{K}_{cell} and $\mathbf{K}(\mathbb{Z})$:

- (1) If $C \in \mathbf{K}_{\text{cell}}$ and $A \in \mathbf{Ch}(\mathbb{Z})$ is acyclic, then every map

$$C \rightarrow A$$

is nullhomotopic.

- (2) If $C \in \mathbf{K}_{\text{cell}}$ and

$$f: A \rightarrow B$$

is a homology equivalence in $\mathbf{Ch}(\mathbb{Z})$, then

$$f_*: \text{hom}_{\mathbf{K}(\mathbb{Z})}(C, A) \xrightarrow{\cong} \text{hom}_{\mathbf{K}(\mathbb{Z})}(C, B)$$

is an isomorphism.

- (3) If $C, D \in \mathbf{K}_{\text{cell}}$ and

$$f: C \rightarrow D$$

is a homology equivalence then f is also a homotopy equivalence.

Our main result is:

Lemma C.1. *Let $F: \mathbf{Ch}(\mathbb{Z}) \rightarrow \text{mod-}\mathbb{Z}$ be a functor such that*

- (1) *whenever $\{C_\alpha\}$ is a direct system of cellular complexes in $\mathbf{Ch}(\mathbb{Z})$,*

$$F(\varinjlim C_\alpha) = \varinjlim F(C_\alpha)$$

- (2) *F factors through the natural quotient $\mathbf{Ch}(\mathbb{Z}) \rightarrow \mathbf{K}_{\text{cell}}$ (i.e., F is a homotopy functor).*

If

$$f: C \rightarrow D$$

is a homology equivalence of nearly free chain-complexes that are bounded from below, then

$$F(f): F(C) \rightarrow F(D)$$

is an isomorphism.

Remark. This essentially says

A homotopy functor that commutes with direct limits is a homology functor of nearly free complexes.

Proof. The conclusion is already known to be true if C and D are in \mathbf{K}_{cell} because then they are homotopy equivalent and F is assumed to be a homotopy functor.

In the general case, let

$$\begin{aligned} C &= \varinjlim C_\alpha \\ D &= \varinjlim D_\alpha \end{aligned}$$

where C_α and D_α are countable chain-complexes. This is possible by proposition 3.3.

In addition, assume $D_\alpha = f(C_\alpha)$ — since homomorphic images of countable complexes are countable. There may be other $D_{\alpha'}$ not in the image of any of the C_α . Our hypotheses imply that

$$\begin{aligned} F(C) &= \varinjlim F(C_\alpha) \\ F(C) &= \varinjlim F(D_\alpha) \end{aligned}$$

Let

$$\begin{aligned} c_\alpha: C_\alpha &\rightarrow C \\ c_{\alpha,\beta}: C_\alpha &\rightarrow C_\beta \\ d_\alpha: D_\alpha &\rightarrow D \\ d_{\alpha,\beta}: D_\alpha &\rightarrow D_\beta \end{aligned}$$

be the inclusions.

The properties of \mathbf{K}_{cell} imply the commutativity of

$$\begin{array}{ccc} \text{(C.1)} & \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\alpha, C) & \xrightarrow{\cong} \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\alpha, D) \\ & \downarrow \text{hom}_{\mathbf{K}(\mathbb{Z})}(f_\alpha, 1) & \downarrow \text{hom}_{\mathbf{K}(\mathbb{Z})}(f_\alpha, 1) \\ & \text{hom}_{\mathbf{K}(\mathbb{Z})}(C_\alpha, C) & \xrightarrow{\cong} \text{hom}_{\mathbf{K}(\mathbb{Z})}(C_\alpha, D) \end{array}$$

in the case where $D_\alpha = f(C_\alpha)$, and

$$\begin{array}{ccc} \text{(C.2)} & \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\beta, C) & \xrightarrow{\cong} \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\beta, D) \\ & \downarrow \text{hom}_{\mathbf{K}(\mathbb{Z})}(d_{\alpha,\beta}, 1) & \downarrow \text{hom}_{\mathbf{K}(\mathbb{Z})}(d_{\alpha,\beta}, 1) \\ & \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\alpha, C) & \xrightarrow{\cong} \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\alpha, D) \end{array}$$

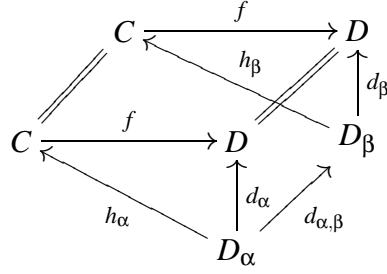
whenever $D_\alpha \subseteq D_\beta$.

Let $h_\alpha \in \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\alpha, C)$ map to $d_\alpha \in \text{hom}_{\mathbf{K}(\mathbb{Z})}(D_\alpha, D)$ under the isomorphism above. They are maps

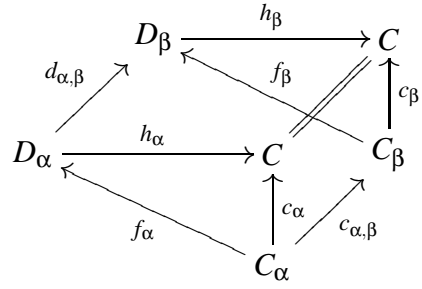
$$h_\alpha: D_\alpha \rightarrow C$$

that are well-defined up to *homotopy*.

Diagrams C.1 and C.2 implies the *homotopy* commutativity of the diagrams



and



whenever $D_\alpha \subseteq D_\beta$.

The fact that F is a homotopy functor implies the *exact* commutativity of the diagrams

(C.3)

Diagram (C.3) is a commutative diagram with nodes $F(C)$, $F(D)$, $F(D_\alpha)$, and $F(D_\beta)$. The nodes $F(C)$ and $F(D)$ are at the top, $F(D_\alpha)$ is at the bottom left, and $F(D_\beta)$ is at the bottom right. There are double arrows between $F(C)$ and $F(D)$ in both directions, labeled $F(f)$. A horizontal arrow points from $F(C)$ to $F(D)$ labeled $F(f)$. A horizontal arrow points from $F(D)$ to $F(C)$ labeled $F(h_\beta)$. A vertical arrow points from $F(D_\alpha)$ to $F(D)$ labeled $F(d_\alpha)$. A vertical arrow points from $F(D_\beta)$ to $F(D)$ labeled $F(d_\beta)$. A diagonal arrow points from $F(D_\alpha)$ to $F(D_\beta)$ labeled $F(d_{\alpha,\beta})$. A diagonal arrow points from $F(D_\alpha)$ to $F(C)$ labeled $F(h_\alpha)$.

and

$$(C.4) \quad \begin{array}{ccccc} & & F(D_\beta) & \xrightarrow{F(h_\beta)} & F(C) \\ & \nearrow^{F(d_{\alpha,\beta})} & & \nwarrow_{F(f_\beta)} & \uparrow^{F(c_\beta)} \\ F(D_\alpha) & \xrightarrow{F(h_\alpha)} & F(C) & & F(C_\beta) \\ & \nwarrow_{F(f_\alpha)} & \uparrow^{F(c_\alpha)} & \nearrow^{F(c_{\alpha,\beta})} & \\ & & F(C_\alpha) & & \end{array}$$

when they are well-defined.

Diagrams C.3 (for all values of α) imply the existence of a map

$$h: \varinjlim F(D_\alpha) = F(D) \rightarrow F(C)$$

that is a right-inverse to $F(f): F(C) \rightarrow F(D)$, and diagrams C.4 imply that it is also a left-inverse. \square

APPENDIX D. PROOF OF LEMMA 5.4

Compare the following definition with definition 3.1 in [11]:

Definition D.1. Let k be 0 or 1. Define $\mathcal{P}(k)$ to be the set of finite sequences $\{u_1, \dots, u_m\}$ of elements each of which is either a \bullet -symbol or an integer $\geq k$.

Given a sequence $\mathbf{u} \in \mathcal{P}(k)$, let $|\mathbf{u}|$ denote the length of the sequence.

Remark D.2. Throughout the rest of this section, we set $k = 0$ if \mathcal{H} is unital and $k = 1$ otherwise.

If $\mathcal{P}_k(n)$ is as defined in definition 3.1 of [11], it is not hard to see that

$$\mathcal{P}(k) = \bigcup_{n=1}^{\infty} \mathcal{P}_k(n)$$

Definition D.3. Let \mathcal{V} be an operad and let $\mathbf{u} = \{u_1, \dots, u_m\} \in \mathcal{P}(k)$, where we impose no condition on k . We define the *generalized composition* with respect to \mathbf{u} , denoted $\gamma_{\mathbf{u}}$, by

$$\begin{aligned} \gamma(\mathbf{u}) &= \circ_{u_m}(\circ_{u_{m-1}} \otimes 1) \cdots (\circ_{u_1} \otimes \cdots \otimes 1) \circ \bigotimes_{\mathbf{u}} \mathbf{1}_j \\ &: \mathcal{V}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}} \mathcal{V}(u_j) \rightarrow \mathcal{V}(\bigsqcup_{i=1}^m u_i) = \mathcal{V}(\mathbf{g}(\mathbf{u})) \end{aligned}$$

where we follow the convention that

$$(1) \ \mathcal{V}(\{\bullet\}) = \mathbb{Z},$$

- (2) $\circ_{\{\bullet\}} = \circ_{\{x\}} \circ (\eta_{\{x\}} \otimes 1): \mathbb{Z} \otimes \mathcal{V}(s) = \mathcal{V}(s) \rightarrow \mathcal{V}(s \setminus \{\bullet\} \sqcup \{x\})$, where $\{x\}$ is a singleton set *not* containing the distinguished element \bullet .

Remark. See definition 2.6 for the definition of $\mathbf{g}(\mathbf{u})$.

If $\mathbf{v} = \{u_{k_1}, \dots, u_{k_l}\} \subset \{u_1, \dots, u_m\}$ is the subset of non- \bullet sets, then $\gamma_{\mathbf{u}}$ is a map

$$\gamma(\mathbf{u}): \mathcal{V}(\mathbf{u}) \otimes \bigotimes_{\mathbf{v}} \mathcal{V}(u_j) \rightarrow \mathcal{V}\left(\bigsqcup_{i=1}^m u_i\right) = \mathcal{V}(\mathbf{g}(\mathbf{u}))$$

If $u \in \mathcal{P}(k)$ with $x \in u$, then $u \sqcup_x x$ represents $(u \setminus x) \sqcup x$ — we have removed x from u and then added the *contents* of x to u . For this notation to make any sense, x must be a set, not an atomic element. Definition D.3 to make any sense, the elements of \mathbf{u} must all be sets and the result of carrying out this operation on all of the elements of \mathbf{u} will be the “flattened form” of \mathbf{u} or $\mathbf{g}(\mathbf{u})$.

Recall that \mathcal{H} is a projective operad with operadic ideal $\mathfrak{I}: \mathcal{J} \hookrightarrow \mathcal{H}$ and K is the kernel of the composite

$$\kappa: [L_{\mathcal{H}}C] \xrightarrow{p} \underline{\mathrm{Hom}}(\mathcal{H}, C^{\otimes}) \xrightarrow{\underline{\mathrm{Hom}}(\mathfrak{I}, 1)} \underline{\mathrm{Hom}}(\mathcal{J}, C^{\otimes})$$

We will show that the coalgebra structure of $L_{\mathcal{H}}C$ induces a coalgebra structure on K that makes it a coalgebra over \mathcal{H}/\mathcal{J} — pulled back over the projection

$$\mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$$

It will then turn out to inherit the “cofreeness” of $L_{\mathcal{H}}C$ as well.

Proposition D.4. *Let $X \in \mathrm{Set}_f$, $x \in X$ and $\{f_y: V_y \rightarrow U_y\}$ be as in definition 2.9. Then*

$$\bigcap_{x \in X} \ker \bigotimes_{X, x} (1, f_x) = \bigotimes_X \ker f_x$$

Proof. The flatness of all the underlying modules implies that

$$\ker \bigotimes_{X, x} (1, f_x) = \bigotimes_{X, x} (V, \ker f_x) = V_{x_1} \otimes \dots \otimes V_{x_k} \otimes \ker f_x \otimes V_{x_{k+2}} \otimes \dots \otimes V_{x_t}$$

and the conclusion follows. \square

Clearly, K inherits a map

$$a: K \rightarrow \underline{\mathrm{Hom}}(\mathcal{H}, C^{\otimes})$$

from its inclusion into $L_{\mathcal{H}}C$. We must show that its image actually lies in

$$\underline{\mathrm{Hom}}(\mathcal{H}, K^{\otimes}) \subseteq \underline{\mathrm{Hom}}(\mathcal{H}, C^{\otimes})$$

We make use of the fact that the structure-map of $L_{\mathcal{H}}C$ is dual to the compositions of the operad \mathcal{H} and that \mathcal{J} is an operadic ideal.

The construction of $L_{\mathcal{H}}C$ in [11] implies that the diagram

$$(D.1) \quad \begin{array}{ccc} & & \underline{\text{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C)^{\otimes}) \\ & \nearrow \alpha & \downarrow y \\ L_{\mathcal{H}}C & \xrightarrow{g} & \prod_{\mathbf{u} \in \mathcal{P}(k)} \text{Hom}_{\mathbb{Z}}(\mathcal{H}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i), C^{\mathfrak{g}(\mathbf{u})}) \end{array}$$

commutes. This is just diagram 3.2 in [11], where:

- (1) α is the adjoint structure map.
- (2) $\kappa: L_{\mathcal{H}}C \hookrightarrow \underline{\text{Hom}}(\mathcal{H}, C^{\otimes})$ is the inclusion (see theorem 3.5).
- (3) $g = \left(\prod_{\mathbf{u} \in \mathcal{P}(k)} c(\mathbf{u}) \right) \circ \kappa$ and the $c(\mathbf{u})$ are defined by

$$\begin{aligned} c(\mathbf{u}) &= \text{Hom}_{\mathbb{Z}}(\gamma(\mathbf{u}), 1): \underline{\text{Hom}}_{\mathbf{u}}(\mathcal{V}(\mathfrak{g}(\mathbf{u})), C^{\mathfrak{g}(\mathbf{u})}) \\ &\rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{V}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}} \mathcal{V}(u_i), C^{\mathfrak{g}(\mathbf{u})}) \end{aligned}$$

— the dual of the generalized structure-map

$$\gamma(\mathbf{u}): \mathcal{V}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}} \mathcal{V}(u_i) \rightarrow \mathcal{V}(\mathfrak{g}(\mathbf{u}))$$

from definition D.3. We assume that $\mathcal{V}(\bullet) = \mathbb{Z}$ and $C^{\bullet} = C$ so that $\underline{\text{Hom}}(\mathcal{V}(\bullet), C^{\bullet}) = C$.

- (4) if $P = \underline{\text{Hom}}(\mathcal{H}, C^{\otimes})$, the map $y = \left(\prod_{\mathbf{u} \in \mathcal{P}(k)} y(\mathbf{u}) \right) \circ \underline{\text{Hom}}(1_{\mathcal{H}}, \kappa^{\otimes})$, where

$$\begin{aligned} y(\mathbf{u}) &= \bar{y}(\mathbf{u})|_{\underline{\text{Hom}}_{\mathbf{u}}(\mathcal{V}(\mathbf{u}), P^{\mathbf{u}})}: \underline{\text{Hom}}_{\mathbf{u}}(\mathcal{V}(\mathbf{u}), P^{\mathbf{u}}) \\ &\rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{V}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}} \mathcal{V}(u_i), C^{\mathfrak{g}(\mathbf{u})}) \end{aligned}$$

and the maps

$$\bar{y}(\mathbf{u}): \text{Hom}_{\mathbb{Z}}(\mathcal{V}(\mathbf{u}), P^{\mathbf{u}}) \rightarrow \text{Hom}_{\mathbb{Z}}(\mathcal{V}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}} \mathcal{V}(u_i), C^{\mathfrak{g}(\mathbf{u})})$$

map the factor

$$\text{Hom}_{\mathbb{Z}}(\mathcal{V}(\mathbf{u}), \bigotimes_{\mathbf{u}} L(u_j)) \subset \text{Hom}_{\mathbb{Z}}(\mathcal{V}(\mathbf{u}), P^{\mathbf{u}})$$

with $L(u_j) = \text{Hom}_{\mathbb{Z}}(\mathcal{V}(u_j), C^{u_j})$ via the map induced by the associativity of the Hom and \otimes functors.

Consider the diagram whose rows are copies of diagram D.1

$$(D.2) \quad \begin{array}{ccccc} K & \xrightarrow{g|K} & W & \xleftarrow{y} & \underline{\text{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C)^\otimes) \\ \downarrow & & \parallel & & \parallel \\ L_{\mathcal{H}}C & \xrightarrow{g} & W & \xleftarrow{y} & \underline{\text{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C)^\otimes) \\ \downarrow \kappa & & \downarrow r_1 & & \downarrow \underline{\text{Hom}}(\mathfrak{t}, 1) \\ \underline{\text{Hom}}(\mathcal{J}, C^\otimes) & \xrightarrow{g_1} & T & \xleftarrow{w} & \underline{\text{Hom}}(\mathcal{J}, (L_{\mathcal{H}}C)^\otimes) \end{array}$$

where

- (1) $W = \prod_{\mathbf{u} \in \mathcal{P}(k)} \text{Hom}_{\mathbb{Z}}(\mathcal{H}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i), C^{\mathfrak{g}(\mathbf{u})})$
- (2) $y: \underline{\text{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C)^\otimes) \rightarrow W$ is defined as in diagram D.1.
- (3) $T = \prod_{\mathbf{u} \in \mathcal{P}(k)} \text{Hom}_{\mathbb{Z}}(\mathcal{J}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i), C^{\mathfrak{g}(\mathbf{u})})$
- (4) The map $g_1 = (\prod_{\mathbf{u} \in \mathcal{P}(k)} \text{Hom}_{\mathbb{Z}}(\tilde{\gamma}(\mathbf{u}), 1)) \circ \kappa$ where

$$\tilde{\gamma}(\mathbf{u}) = \gamma(\mathbf{u})|_{\mathcal{J}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i)}: \mathcal{J}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i) \rightarrow \mathcal{H}(\mathfrak{g}(\mathbf{u}))$$

- (5) The map $r_1 = \text{Hom}_{\mathbb{Z}}(j_1, 1)$ where

$$j_1: \mathcal{J}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i) \hookrightarrow \mathcal{H}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i)$$

for $\mathbf{u} \in \mathcal{P}(k)$, are the inclusions.

- (6) $\mathfrak{t}: \mathcal{J} \hookrightarrow \mathcal{H}$ is the inclusion.

Suppose $r \in K$. Then the image of r under the downward maps on the left of diagram D.2 must be 0, since K is the kernel of κ . On the other hand, $r = y(h)$ in the top two rows of this diagram.

The commutativity of diagram D.2 implies that the image of h under the downward maps on the *right* is also 0, so that the coproduct of r in the kernel of $\underline{\text{Hom}}(\mathfrak{t}, 1)$. This implies that the coproduct of K is the pullback of a map

$$K \rightarrow \underline{\text{Hom}}(\mathcal{H}/\mathcal{J}, C^\otimes)$$

over the projection $p: \mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$.

Let $X \in \text{Set}_f$ and let $x \in X$ be an arbitrary element. We claim that the diagram

$$(D.3) \quad \begin{array}{ccccc} K & \xrightarrow{g|K} & W' & \xleftarrow{y} & \underline{\mathrm{Hom}}(\mathcal{H}/\mathcal{J}, (L_{\mathcal{H}}C)^{\otimes}) \\ \downarrow & & \downarrow \mathrm{Hom}_{\mathbb{Z}}(p \otimes 1, 1) & & \downarrow \underline{\mathrm{Hom}}(p, 1) \\ L_{\mathcal{H}}C & \xrightarrow{g} & W & \xleftarrow{y} & \underline{\mathrm{Hom}}(\mathcal{H}, (L_{\mathcal{H}}C)^{\otimes}) \\ \downarrow \kappa & & \downarrow p_{W(X)} & & \downarrow p_X \\ \underline{\mathrm{Hom}}(\mathcal{H}, C^{\otimes}) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\gamma, 1)} & W(X) & \xleftarrow{y(X)} & \underline{\mathrm{Hom}}_X(\mathcal{H}(X), (L_{\mathcal{H}}C)^X) \\ \downarrow \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}, 1) & & \downarrow \theta(X, x) & & \downarrow \phi(X, x) \\ \underline{\mathrm{Hom}}(\mathcal{J}, C^{\otimes}) & \xrightarrow{\mathrm{Hom}_{\mathbb{Z}}(\gamma, 1)} & Y(X, x) & \xleftarrow{y(X, x)} & \underline{\mathrm{Hom}}_X(\mathcal{H}(X), M(X, x)) \end{array}$$

commutes, where

(1) p_X and $p_{W(X)}$ are projections onto direct factors.

(2) $W' = \prod_{\mathbf{u} \in \mathcal{P}(k)} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}(\mathbf{u})/\mathcal{J}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i), C^{\mathfrak{g}(\mathbf{u})})$

(3) $W(X) = \prod_{\substack{\mathbf{u} \in \mathcal{P}(k) \\ s: X \rightarrow \mathfrak{f}(\mathbf{u})}} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}(\mathbf{u}) \otimes \bigotimes_{u_i \in \mathbf{u}} \mathcal{H}(u_i), C^{\mathfrak{g}(\mathbf{u})}), \quad \text{where}$

$s: X \rightarrow \mathfrak{f}(\mathbf{u})$ is a set-bijection. This is exactly like W , except that we only consider \mathbf{u} such that $\mathfrak{f}(\mathbf{u})$ has the same *cardinality* as the set X .

(4) $Y(X, x) = \prod_{\substack{\mathbf{u} \in \mathcal{P}(k) \\ s: X \rightarrow \mathfrak{f}(\mathbf{u})}} \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}, s(x)} (\mathcal{H}, \mathcal{J}), C^{\mathfrak{g}(\mathbf{u})}), \text{ where } x \in X \text{ is}$

any element — see definition 2.9. This is exactly like $W(X)$, except that the $s(x)^{\mathrm{th}}$ factor of $\mathcal{H}(u_i)$ has been replaced with $\mathcal{J}(s(x))$.

(5) $\theta(X, x) = \mathrm{Hom}_{\mathbb{Z}}(1 \otimes \bigotimes_{X, x} (1, \mathfrak{t}), 1): W(X) \rightarrow Y(X, x)$. This is the dual

of $1 \otimes \bigotimes_{X, x} (1, \mathfrak{t})$, which is the identity, except for the x^{th} factor on the right. For this factor it is the inclusion $\mathfrak{t}: \mathcal{J}(s(x)) \hookrightarrow \mathcal{H}(s(x))$.

(6) $\phi(X, x) = \mathrm{Hom}_{\mathbb{Z}} \left(1, \bigotimes_{X, x} (1, \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{t}, 1)) \right)$

(7) $M(X, x) = \bigotimes_{X, x} (\underline{\mathrm{Hom}}(\mathcal{H}, C^{\otimes}), \underline{\mathrm{Hom}}(\mathcal{J}, C^{\otimes}))$ — see definition 2.9.

This is the similar to $(\underline{\mathrm{Hom}}(\mathcal{H}, C^{\otimes}))^X$, except that the x^{th} factor has been replaced by $\underline{\mathrm{Hom}}(\mathcal{J}, C^{\otimes})$.

(8) $y(X)$ is defined as $y(\mathbf{u})$ in diagram D.1 and $y(X, x)$ is defined analogously — with the x^{th} factor mapping $\underline{\mathrm{Hom}}(\mathcal{J}, C^{\otimes})$.

The upper squares of diagram D.3 commute because they did in diagram D.2.

The lower left square of diagram D.3 commutes because it is the dual of the diagram

$$\begin{array}{ccc} \mathcal{H}(\mathfrak{g}(\mathbf{u})) & \xleftarrow{\gamma} & \mathcal{H}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}} \mathcal{H}(u_i) \\ \uparrow \mathfrak{t} & & \uparrow 1 \otimes \bigotimes_{\mathbf{u}, s(x)} (1, \mathfrak{t}) \\ \mathcal{J}(\mathfrak{g}(\mathbf{u})) & \xleftarrow{\gamma} & \mathcal{H}(\mathbf{u}) \otimes \bigotimes_{\mathbf{u}, s(x)} (\mathcal{H}, \mathcal{J}) \end{array}$$

which is well-defined and commutes because \mathcal{J} is an operadic ideal of \mathcal{H} . The lower right square of diagram D.3 commutes because of the naturality of the γ -maps.

A diagram-chase around the outer rim of diagram D.3 shows that if $k \in K$, then the coproduct of k , evaluated on any element of $\mathcal{H}(X)/\mathcal{J}(X)$ (or $\mathcal{H}(X)$) gives a result that lies in the kernel of $\bigotimes_{X, x} (1, \text{Hom}_{\mathbb{Z}}(\mathfrak{t}, 1))$ for any finite set X and any element $x \in X$, hence is in K^X — see proposition D.4. It follows that K is a sub-coalgebra of $L_{\mathcal{H}}C$ and one that has been pulled back from \mathcal{H}/\mathcal{J} .

The lemma's final statement follows from the universal property of cofree coalgebras. Suppose M is any coalgebra over \mathcal{H}/\mathcal{J} equipped with a chain-map $\alpha: M \rightarrow C$. By composition with the projection $p: \mathcal{H} \rightarrow \mathcal{H}/\mathcal{J}$, we may regard M as a coalgebra over \mathcal{H} . The universal property of a cofree coalgebra implies that there exists a *unique* morphism of \mathcal{H} -coalgebras

$$M \rightarrow L_{\mathcal{H}}C$$

that makes the diagram

$$\begin{array}{ccc} M & \longrightarrow & L_{\mathcal{H}}C \\ & \searrow \alpha & \downarrow \varepsilon \\ & & C \end{array}$$

commute (where $\varepsilon: L_{\mathcal{H}}C \rightarrow C$ is the cogeneration map). But the image of M must lie in $K \subseteq L_{\mathcal{H}}C$, hence K has the universal property of a cofree coalgebra over \mathcal{H}/\mathcal{J} .

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